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Parabolic equations with the second-order Cauchy conditions on the boundary

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Abstract
The paper studies some ill-posed boundary value problems on semi-plane for parabolic equations with the homogenous Cauchy condition at initial time and with the second-order Cauchy condition on the boundary of the semi-plane. A class of inputs that allows some regularity is suggested and described explicitly in the frequency domain. This class is everywhere dense in the space of square integrable functions.

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Parabolic equations such as heat equations have fundamental significance for natural sciences, and various boundary value problems for them were widely studied including well-posed problems as well as the so-called ill-posed problems that are often significant for applications. The present paper introduces and investigates a special boundary value problem on semi-plane for parabolic equations with the homogenous Cauchy condition at initial time and with the second-order Cauchy condition on the boundary of the semi-plane. The problem is ill posed. A set of solvability or a class of inputs that allows some regularity in the form of prior energy-type estimates is suggested and described explicitly in the frequency domain. This class is everywhere dense in the class of L_2-integrable functions. This result looks counterintuitive, since these boundary conditions are unusual; solvability of this boundary value problem for a wider class of inputs is inconsistent with basic theory.

1. The problem setting

Let us consider the following boundary value problem:

$$\begin{aligned} &a\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + b\frac{\partial u}{\partial x}(x,t) + cu(x,t) + f(x,t),\\ &u(x,0) \equiv 0, \\ &u(0,t) \equiv g_0(t), \qquad \frac{\partial u}{\partial x}(0,t) \equiv g_1(t). \end{aligned} \tag{1}$$

Here, $x > 0, t > 0$ and $a > 0$; $b, c \in \mathbf{R}$ are constants; $g_k \in L_2(0, +\infty)$, $k = 1, 2$; and f is a measurable function such that $\int_0^y \mathrm{d}x \int_0^\infty |f(x,t)|^2 \, \mathrm{d}t < +\infty$ for all $y > 0$.

This problem is ill posed (see Tikhonov and Arsenin (1977)).

Let $\mu \triangleq b^2/4 - c$. We assume that $\mu > 0$. Note that this assumption does not reduce generality for the cases when we are interested in a solution on a finite time interval, since we can rewrite the parabolic equation as that with c replaced by $c - M$ for any $M > 0$ and $g_k(t)$ replaced by $\mathrm{e}^{-Mt} g_k(t)$; the solution u_M of the new equation is related to the solution u of the old one as $u_M(x,t) = \mathrm{e}^{-Mt} u(x,t)$.

Definitions and special functions

Let $\mathbf{R}^+ \triangleq [0, +\infty)$, $\mathbf{C}^+ \triangleq \{z \in \mathbf{C} : \mathrm{Re}\, z > 0\}$. For $v \in L_2(\mathbf{R})$, we denote by $\mathcal{F}v$ and $\mathcal{L}v$ the Fourier and the Laplace transforms, respectively:

$$V(\mathrm{i}\omega) = (\mathcal{F}v)(\mathrm{i}\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}\omega t} v(t) \, \mathrm{d}t, \qquad \omega \in \mathbf{R}, \tag{2}$$

$$V(p) = (\mathcal{L}v)(p) \triangleq \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{e}^{-pt} v(t) \, \mathrm{d}t, \qquad p \in \mathbf{C}^+. \tag{3}$$

Let H^r be the Hardy space of holomorphic on $\mathbf{C}^+$ functions $h(p)$ with finite norm $\|h\|_{H^r} = \sup_{k>0} \|h(k + \mathrm{i}\omega)\|_{L_r(\mathbf{R})}$, $r \in [1, +\infty]$ (see, e.g., Duren (1970)).

For $y > 0$, let $\mathcal{W}(y)$ be the Banach space of the functions $u : (0, y) \times \mathbf{R}^+ \to \mathbf{R}$ with the finite norm

$$\|u\|_{\mathcal{W}(y)} \triangleq \sup_{x \in (0,y)} \left(\|u(x,\cdot)\|_{L_2(\mathbf{R}^+)} + \left\| \frac{\partial u}{\partial x}(x,\cdot) \right\|_{L_2(\mathbf{R}^+)} + \left\| \frac{\partial^2 u}{\partial x^2}(x,\cdot) \right\|_{L_2(\mathbf{R}^+)} + \left\| \frac{\partial u}{\partial t}(x,\cdot) \right\|_{L_2(\mathbf{R}^+)} \right).$$

The class $\mathcal{W}(y)$ is such that all the equations presented in problem (1) are well defined for any $u \in \mathcal{W}(y)$ and in the domain $(0, y) \times \mathbf{R}^+$. For instance, if $v \in \mathcal{W}(y)$, then, for any $t_* > 0$, we have that $v|_{[0,y]\times[0,t_*]} \in C([0,t_*], L_2(0,y))$ as a function of $t \in [0, t_*]$. Hence, the initial condition at time $t = 0$ is well defined as an equality in $L_2([0, y])$. Further, we have that $v|_{[0,y]\times\mathbf{R}^+} \in C([0,y], L_2(\mathbf{R}^+))$ and $\frac{\partial v}{\partial x}\big|_{[0,y]\times\mathbf{R}^+} \in C([0,y], L_2(\mathbf{R}^+))$ as functions of $x \in [0, y]$. Hence the functions $v(0,t)$, $\frac{\mathrm{d}v}{\mathrm{d}x}(x,t)|_{x=0}$ are well defined as elements of $L_2(\mathbf{R}^+)$, and the boundary value conditions at $x = 0$ are well defined as equalities in $L_2(\mathbf{R}^+)$.

Special smoothing kernel

Let us introduce the set of the following special function:

$$K(p) = K_{\alpha,\beta,q}(p) \triangleq \mathrm{e}^{-\alpha(p+\beta)^q}, \qquad p \in \mathbf{C}^+. \tag{4}$$

Here $\alpha > 0, \beta > 0$ are reals and $q \in \left(\frac{1}{2}, 1\right)$ is a rational number. We mean the branch of $(p + \beta)^q$ such that its argument is $q \,\mathrm{Arg}\,(p + \beta)$, where $\mathrm{Arg}\, z \in (-\pi, \pi]$ denotes the principal value of the argument of $z \in \mathbf{C}$.

The functions $K_{\alpha,\beta,q}(p)$ are holomorphic in $\mathbf{C}^+$, and

$$\ln |K(p)| = -\mathrm{Re}\,(\alpha(p+\beta)^q) = -\alpha|p+\beta|^q \cos[q \,\mathrm{Arg}\,(p+\beta)].$$

In addition, there exists $M = M(\beta, q) > 0$ such that $\cos[q \,\mathrm{Arg}\,(p + \beta)] > M$ for all $p \in \mathbf{C}^+$. It follows that

$$|K(p)| \leqslant \mathrm{e}^{-\alpha M|p+\beta|^q} < 1, \qquad p \in \mathbf{C}^+. \tag{5}$$

Hence, $K \in H^r$ for all $r \in [1, +\infty]$.

Proposition 1. *Let $\beta > 0$ and a rational number $q \in \left(\frac{1}{2}, 1\right)$ be given. Let $v \in L_2(\mathbf{R}^+)$, $V = \mathcal{L}v \in H^2$. For $\alpha > 0$, set $V_\alpha \stackrel{\Delta}{=} K_{\alpha,\beta,q}V$, $v_\alpha \stackrel{\Delta}{=} \mathcal{F}^{-1}V_\alpha(\mathrm{i}\omega)|_{\omega\in\mathbf{R}}$. Then $V_\alpha \in H^2$ and $v_\alpha \to v$ in $L_2(\mathbf{R}^+)$ as $\alpha \to 0, \alpha > 0$.*

Proof. Clearly, $V_\alpha(\mathrm{i}\omega) \to V(\mathrm{i}\omega)$ as $\alpha \to 0$ for a.e. $\omega \in \mathbf{R}$. By (4), $V_\alpha \in H^2$. In addition, $|K_{\alpha,\beta,q}(\mathrm{i}\omega)| \leqslant 1$. Hence, $|V_\alpha(\mathrm{i}\omega) - V(\mathrm{i}\omega)| \leqslant 2|V(\mathrm{i}\omega)|$. We have that $\|V(\mathrm{i}\omega)\|_{L_2(\mathbf{R})} = \|v\|_{L_2(\mathbf{R}^+)} < +\infty$. By the Lebesgue dominance theorem, it follows that

$$\|V_\alpha(\mathrm{i}\omega) - V(\mathrm{i}\omega)\|_{L_2(\mathbf{R})} \to 0 \qquad \text{as} \qquad \alpha \to 0.$$

Hence, $v_\alpha \to v$ in $L_2(\mathbf{R}^+)$ as $\alpha \to 0$. Then the proof follows. □

The inverse Fourier transform $k(t) = \mathcal{F}^{-1}K_{\alpha,\beta,q}(\mathrm{i}\omega)|_{\omega\in\mathbf{R}}$ can be viewed as a smoothing kernel; $k(t) = 0$ for $t < 0$. It can be seen that k has derivatives of any order.

Denote by $\mathcal{C}$ the set of functions $v : \mathbf{R}^+ \to \mathbf{R}$ such that there exist $\alpha > 0$, $\beta > 0$, and a rational number $q \in \left(\frac{1}{2}, 1\right)$, such that $\widehat{V} \in H^2$, where $\widehat{V}(p) = K_{\alpha,\beta,q}(p)^{-1}V(p)$, $V = \mathcal{L}v$.

The set $\mathcal{C}$ includes outputs of the convolution integral operators with the kernels $k(t)$. By proposition 1, it follows that the set $\mathcal{C}$ is everywhere dense in $L_2(\mathbf{R}^+)$.

2. The main result

Set $F(x,\cdot) \stackrel{\Delta}{=} \mathcal{L}f(x,\cdot)$, where $x > 0$ is given, and $G_k \stackrel{\Delta}{=} \mathcal{L}g_k$, $k = 0, 1$.

Theorem 1. *Let the functions f and g_k are such that there exist $y > 0, \alpha > 0, \beta > 0$, a rational number $q \in \left(\frac{1}{2}, 1\right)$, such that $\widehat{G}_k \in H^2, \widehat{F}(x,\cdot) \in H^2$ for a.e. $x > 0$ and $\int_0^y \|\widehat{F}(s,\cdot)\|_{H^2}\,\mathrm{d}s < +\infty$, where*

$$\widehat{F}(x,p) \stackrel{\Delta}{=} \frac{F(x,p)}{K(p)}, \qquad \widehat{G}_k(p) \stackrel{\Delta}{=} \frac{G_k(p)}{K(p)}, \tag{6}$$

and where the function $K = K_{\alpha,\beta,q}$ is defined by (4) (in particular, this means that $g_k \in \mathcal{C}$ and $f(x,\cdot) \in \mathcal{C}$ for a.e. $x \in [0, y]$). Then there exists an unique solution $u(x,t)$ of problem (1) in the domain $(0, y) \times \mathbf{R}^+$ in the class $\mathcal{W}(y)$. Moreover, there exists a constant $C(y) = C(a, b, c, \alpha, \beta, q, y)$ such that

$$\|u\|_{\mathcal{W}(y)} \leqslant C(y)\left(\|\widehat{G}_1\|_{H^2} + \|\widehat{G}_2\|_{H^2} + \int_0^x \|\widehat{F}(s,\cdot)\|_{H^2}\,\mathrm{d}s\right).$$

Remark 1. Theorem 1 requires that functions f and g_k are smooth in t; in particular, they belong to C^∞ in t. However, it is not required that $f(x,t)$ is smooth in x.

Proof of theorem 1. Instead of (1), consider the following problems for $p \in \mathbf{C}^+$:

$$\begin{aligned} &apU(x,p) = \frac{\partial^2 U}{\partial x^2}(x,p) + b\frac{\partial U}{\partial x}(x,p) + cU(x,p) + F(x,p), \qquad x > 0,\\ &U(0,p) \equiv G_0(p), \qquad \frac{\partial U}{\partial x}(0,p) \equiv G_1(p). \end{aligned} \tag{7}$$

Let $\lambda_k = \lambda_k(p)$ be the roots of the equation $\lambda^2 + b\lambda + (c - ap) = 0$. Clearly, $\lambda_{1,2} \stackrel{\Delta}{=} -b/2 \pm \sqrt{ap + \mu}$. Recall that $\mu > 0$. It follows that the functions $(\lambda_1(p) - \lambda_2(p))^{-1}$ and $\lambda_k(p)(\lambda_1(p) - \lambda_2(p))^{-1}$, $k = 1, 2$, belong to H^∞.

For $x \in (0, y]$, the solution of (7) is

$$U(x,p) = \frac{1}{\lambda_1 - \lambda_2}\Bigg((G_1(p) - \lambda_2 G_0(p))\,\mathrm{e}^{\lambda_1 x} - (G_1(p) - \lambda_1 G_0(p))\,\mathrm{e}^{\lambda_2 x} - \int_0^x \mathrm{e}^{\lambda_1(x-s)} F(s,p)\,\mathrm{d}s + \int_0^x \mathrm{e}^{\lambda_2(x-s)} F(s,p)\,\mathrm{d}s \Bigg). \tag{8}$$

This can be derived, for instance, using the Laplace transform method applied to the linear ordinary differential equation (7), and having in mind that

$$\frac{1}{\lambda^2 + b\lambda + c - ap} = \frac{1}{(\lambda-\lambda_1)(\lambda-\lambda_2)} = \frac{1}{\lambda_1-\lambda_2}\left(\frac{1}{\lambda-\lambda_1} - \frac{1}{\lambda-\lambda_2}\right),$$

$$\frac{\lambda}{\lambda^2 + b\lambda + c - ap} = \frac{\lambda}{(\lambda-\lambda_1)(\lambda-\lambda_2)} = \frac{1}{\lambda_1-\lambda_2}\left(\frac{\lambda_1}{\lambda-\lambda_1} - \frac{\lambda_2}{\lambda-\lambda_2}\right).$$

Let $x \in (0, y)$, $s \in [0, x]$. The functions $\mathrm{e}^{(x-s)\lambda_k(p)}$, $k = 1, 2$, are holomorphic in $\mathbf{C}^{+}$. We have

$$\ln|\,\mathrm{e}^{(x-s)\lambda_k(p)}| = \mathrm{Re}\,((x-s)\lambda_k(p)) = (x-s)\left(-\frac{b}{2} \pm |ap+\mu|^{1/2}\cos\frac{\mathrm{Arg}\,(ap+\mu)}{2}\right),$$

where $k = 1, 2$, $p \in \mathbf{C}^{+}$. It follows that

$$|K(p)\,\mathrm{e}^{(x-s)\lambda_k(p)}| \leqslant \mathrm{e}^{(x-s)[-b/2+|ap+\mu|^{1/2}]-\alpha M|p+\beta|^q},$$

$k = 1, 2$, $p \in \mathbf{C}^{+}$. Similarly,

$$|K(p)\,\mathrm{e}^{\lambda_k x}| \leqslant \mathrm{e}^{x[-b/2+|ap+\mu|^{1/2}]-\alpha M|p+\beta|^q}.$$

Since $q > 1/2$, it follows that $K(p)\,\mathrm{e}^{\lambda_k x} \in H^r$, $K(p)\,\mathrm{e}^{(x-s)\lambda_k(p)} \in H^r$, $pK(p)\,\mathrm{e}^{\lambda_k x} \in H^r$ and $pK(p)\,\mathrm{e}^{(x-s)\lambda_k(p)} \in H^r$, for $r = 2$ and $r = +\infty$. Moreover, we have

$$\sup_{s\in[0,x]} \|p^m\,\mathrm{e}^{\lambda_k(p)s} G_k(p)\|_{H^2} \leqslant C_1(x)\|\widetilde{G}_k\|_{H^2},$$

$$\sup_{s\in[0,x]} \|p^m\,\mathrm{e}^{\lambda_k(p)s} K(p)\|_{H^\infty} \leqslant C_2(x),$$

where $m = 0, 1$. Hence,

$$\sup_{x\in[0,y]} \left\| p^m \int_0^x \mathrm{e}^{(x-s)\lambda_k} F(s,p)\,\mathrm{d}s \right\|_{H^2} \leqslant \sup_{x\in[0,y]} \int_0^x \left\| \mathrm{e}^{(x-s)\lambda_k} p^m F(s,p) \right\|_{H^2} \mathrm{d}s$$
$$\leqslant \sup_{x\in[0,y]} \int_0^x \|p^m\,\mathrm{e}^{\lambda_k(x-s)} K(s)\|_{H^\infty} \|\widetilde{F}(s,p)\|_{H^2}\,\mathrm{d}s \leqslant C_2(y) \int_0^y \|\widehat{F}(s,p)\|_{H^2}\,\mathrm{d}s,$$

where $m = 0, 1$. Here $C_1(x)$, $C_2(x)$ are constants that depend on $a, b, c, \alpha, \beta, q, x$. It follows that $p^m\,\mathrm{e}^{\lambda_k x} G_m(p) \in H^2$ and $p^m \int_0^x \mathrm{e}^{(x-s)\lambda_k} F(p,s)\,\mathrm{d}s \in H^2$ for any $x > 0$, $m = 0, 1$, $k = 1, 2$.

Recall that $\lambda_k = \lambda_k(p)$. Let

$$N \triangleq \left\| \frac{1}{\lambda_1 - \lambda_2} \right\|_{H^\infty} + \sum_{k=1,2} \left\| \frac{\lambda_k}{\lambda_1 - \lambda_2} \right\|_{H^\infty}.$$

It follows from the above estimates that

$$\|p^m U(x,p)\|_{H^2} \leqslant N\left(C_1(y) \sum_{k=1,2} \left\|\widehat{G}_k\right\|_{H^2} + C_2(y) \int_0^x \|\widehat{F}(s,p)\|_{H^2}\mathrm{d}s \right), \qquad m = 0, 1. \tag{9}$$

It follows that the corresponding inverse Fourier transforms $u(x,\cdot) = \mathcal{F}^{-1}U(x,\mathrm{i}\omega)|_{\omega\in\mathbf{R}}$, $\frac{\partial u}{\partial t}(x,\cdot) = \mathcal{F}^{-1}(pU(x,\mathrm{i}\omega)|_{\omega\in\mathbf{R}})$ are well defined and are vanishing for $t<0$. In addition, we have that $\overline{U(x,\mathrm{i}\omega)} = U(x,-\mathrm{i}\omega)$ (for instance, $\overline{K(\mathrm{i}\omega)} = K(-\mathrm{i}\omega)$, $\overline{\mathrm{e}^{(x-s)\lambda_k(\mathrm{i}\omega)}} = \mathrm{e}^{(x-s)\lambda_k(-\mathrm{i}\omega)}$, etc). It follows that the inverse of the Fourier transform $u(x,\cdot) = \mathcal{F}^{-1}U(x,\cdot)$ is real.

Further, we have that

$$\frac{\partial U}{\partial x}(x,p) = \frac{1}{\lambda_1-\lambda_2}\biggl((G_1(p)-\lambda_2 G_0(p))\lambda_1\,\mathrm{e}^{\lambda_1 x} - (G_1(p)-\lambda_1 G_0(p))\lambda_2\,\mathrm{e}^{\lambda_2 x}$$
$$-\lambda_1\int_0^x \mathrm{e}^{\lambda_1(x-s)}F(s,p)\,\mathrm{d}s + \lambda_2\int_0^x \mathrm{e}^{\lambda_2(x-s)}F(s,p)\,\mathrm{d}s\biggr). \tag{10}$$

Since $\lambda_1(p)\lambda_2(p) = c - ap$, we again obtain that

$$\left\|\frac{\partial U}{\partial x}(x,p)\right\|_{H^2} \leqslant C_3(y)\left(\sum_{k=1,2}\|\widehat{G}_k\|_{H^2} + \int_0^x \|\widehat{F}(s,p)\|_{H^2}\,\mathrm{d}s\right). \tag{11}$$

By (7), $\partial^2U/\partial x^2$ can be expressed as a linear combination of $F, G_k, U, pU, \partial U/\partial x$. By (9)–(11),

$$\left\|\frac{\partial^2 U}{\partial x^2}(x,p)\right\|_{H^2} \leqslant C_4(y)\left(\left\|\frac{\partial U}{\partial x}(x,p)\right\|_{H^2} + \sum_{m=0,1}\|p^m U(x,p)\|_{H^2} + \|F(x,p)\|_{H^2}\right).$$

We have that $|K(p)| < 1$ on C^+ and $\|F(s,p)\|_{H^2} \leqslant \|\widehat{F}(s,p)\|_{H^2}$. It follows that

$$\left\|\frac{\partial^2 U}{\partial x^2}(x,p)\right\|_{H^2} \leqslant C_5(y)\left(\sum_{k=1,2}\|\widehat{G}_k\|_{H^2} + \int_0^x \|\widehat{F}(s,p)\|_{H^2}\,\mathrm{d}s\right). \tag{12}$$

Here $C_k(y)$ are constants that depend on $a, b, c, \alpha, \beta, q, y$. By (9)–(12), estimate (6) holds.

Therefore, $u(x,\cdot) = \mathcal{F}^{-1}U(x,\mathrm{i}\omega)|_{\omega\in\mathbf{R}}$ is the solution of (1) in $\mathcal{W}(y)$. The uniqueness is ensured by the linearity of the problem, by estimate (6), and by the fact that $\mathcal{L}u(x,\cdot)$, $\mathcal{L}(\partial^k u(x,\cdot)/\partial x^k)$ and $\mathcal{L}(\partial u(x,\cdot/\partial t)$ are well defined on $\mathbf{C}^+$ for any $u \in \mathcal{W}(y)$. This completes the proof of theorem 1. □

Remark 2. It can be seen from the proof that it is crucial that $u(x,0)\equiv 0$. Non-zero initial conditions cannot be included.

References

Duren P 1970 *Theory of H^p-Spaces* (New York: Academic)
Tikhonov A N and Arsenin V Y 1977 *Solutions of Ill-Posed Problems* (Washington, DC: Winston)